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Convergence of Discrete Rational Approximations

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INTRODUCTION

In this paper the approximation of a continuous function by generalized rational functions over finite subsets of some interval $[a, b]$ of the real line will be considered. Our principal purpose is to extend a convergence result proved in [1] for polynomial rational functions to a more general class of rational families that would include the polynomial and trigonometric rational families as examples. In particular, we shall be interested in the question of the existence and of the convergence of these discrete best approximations to a best approximation over $[a, b]$ as the size and density of the finite set increases. Under certain circumstances (which always hold in the polynomial or trigonometric case) a subsequence will converge uniformly over $[a, b]$ to a best approximation (over $[a, b]$).

This result will be obtained using the results and techniques of both [1] and [2]. As in [1] and [2] we shall consider only approximation in the L_t norms for $1 \leqslant t < \infty$.

Let $P = \text{span}\{\rho_1, ..., \rho_n\}$ and $Q = \text{span}\{\sigma_1, ..., \sigma_m\}$ be Haar subspaces of $\mathscr{C}[a, b]$ of dimension n and m respectively and let $1 \leqslant t < \infty$ be arbitrary but fixed. There are then three rational families that come into consideration.

$$R^- = \{p/q \mid p \in P, q \in Q, q(x) > 0 \text{ for all } x \in [a, b]\}$$
$$R = \{p/q \mid p \in P, q \in Q, q(x) \geqslant 0 \text{ for all } x \in [a, b] \text{ and } \| p/q \|_t < \infty\}$$
$$\hat{R} = \{p/q \mid p \in P, q \in Q, \| p/q \|_t < \infty\}.$$

In [1] it was shown that R is the norm closure of R^- and that each $f \in L_t[a, b]$ has a best approximation in R. For this paper it is convenient to consider the larger class $\hat{R}$. Using the same techniques exactly as in [2] we have the following theorem which we only state.

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THEOREM 1. *Let $f \in L_t[a, b]$ where $1 \leqslant t < \infty$ is arbitrary but fixed. Then f has a best approximation in $\hat{R}$.*

The above result is the only one from [2] that will be needed in this paper.

DISCRETE PROBLEM

Let $X = \{x_1, ..., x_M\} \subset [a, b]$ with $M \geqslant m + n + 1$ and let $R(X)$ denote the set $\{p/q \mid p \in P, q \in Q, q(x) \neq 0 \text{ for all } x \in X\}$. Then if $\|\cdot\|$ is a given norm on $B(X) = \{f \mid f \text{ is a real-valued function on } X\}$ and $f \in B(X)$ is given, we seek $r^* \in R(X)$ such that $\|f - r^*\| = \inf_{r \in R(X)} \|f - r\|$. In [1] the existence question using norms of the type $\|f\| = [\sum_{x \in X} |f(x)|^t]^{1/t}$ for $1 \leqslant t < \infty$ was studied. Since X has finitely many elements, best approximations always exist in the pointwise closure of $R(X)$ (denoted by $\overline{R(X)}$). Thus an explicit description of $\overline{R(X)}$ is of interest. This was done in [1] and we now list those results since they will be needed here. As in [1], the notation $R(Y)$ where Y is some subset of $[a, b]$ will denote the set $\{p/q \mid p \in P, q \in Q, q(x) \neq 0 \text{ for all } x \in Y\}$.

DEFINITION. Let S_1 denote the set of functions g in $B(X)$ such that there exists some set $S \subset X$ (depending on g) containing at most $k = \min(n - 1, m - 1)$ elements and some rational function p/q in $R(X \sim S)$ with $p(x) = q(x) = 0$ for all $x \in S$ for which $g = p/q$ on $X \sim S$.

DEFINITION. Let S_2 denote the set of all functions g in $B(X)$ such that g is zero except precisely on some subset $T \subset X$ (depending on g) having at most $m - 1$ elements.

We then have the following from [1]:

THEOREM 2. *The set $\overline{R(X)}$ is given by $S_1 \cup S_2$.*

COROLLARY. *Let $1 \leqslant t < \infty$ be arbitrary. If $f \in B(X)$ has $g \in S_1$ (or $g \in S_2$) as a best approximation with respect to the corresponding discrete L_t norm then $g = f$ on the associated set S (or T if $g \in S_2$).*

CONVERGENCE OF DISCRETE APPROXIMATIONS

Now assume that $f \in C[a, b]$ is given and that it is desired to calculate a best approximation to f from R^+ with respect to the L_t norm where $1 \leqslant t < \infty$ is arbitrary but fixed. To do this, $[a, b]$ is replaced by a sequence of grids of the form $[h_v] = \{a + kh_v \mid k = 0, 1,..., N_v\}$ where $h_v = (b - a)/N_v$ and $N_v \to \infty$ as $v \to \infty$ and the L_t norm is replaced by its discrete analog,

$$\| g \|_{[h_v]} = h_v^{1/t} \left(\sum_{x \in [h_v]} | g(x)|^t \right)^{1/t}.$$

A best approximation is calculated with respect to each of the above discrete norms from $R([h_v])$ (provided such a best approximation exists) and the convergence question is whether or not these computed approximations converge in some sense to a best approximation to f over $[a, b]$ from R^+ (or possibly R or $\hat{R}$). The following concept of normality is basic to the question of convergence.

DEFINITION. An element $r_0 = p_0/q_0 \in \hat{R}$ is called normal if $\dim(p_0Q + q_0P) = m + n - 1$. The symbol NP will denote the set of all functions in $L_t[a, b]$ having only normal best approximations in $\hat{R}$. (Recall t is arbitrary but fixed.)

The following result from [3] will be useful.

THEOREM 3. *Let* $[\alpha, \beta]$ *be a subinterval of* $[a, b]$ *and define a norm on* Q *(restricted to* $[\alpha, \beta]$*) by* $\| q \| = \sum_{k=1}^m | b_k |$ *where* $q = \sum_{k=1}^m b_k\sigma_k$. *Assume that the set* $Q^+ = \{q \in Q \mid q(x) > 0$ *for all* $x \in [\alpha, \beta]$ *and* $\| q \| = 1\}$ *is non-empty. Let* $R^+ = \{p/q \mid p \in P, q \in Q^+\}$. *Define* $A\colon P \oplus Q^+ \to R^+$ *by* $A(p, q) = p/q$. *Then* A *is topological at* (p_0, q_0) *if and only if* p_0/q_0 *is normal.*

Remark. In [3] the norm used on the elements of Q was the uniform norm. The proof, however, is the same using the norm given above and this is more convenient for the purposes of this paper.

In what follows, the symbol $\| g \|_A$ where A is a subset of $[h_v]$ and $g \in B([h_v])$ will denote $h^{1/t}(\sum_{x \in A} | g(x)|^t)^{1/t}$. To simplify notation we will shorten $\| \ \|_{[h_v]}$ to $\| \ \|_v$. The L_t norm on $[a, b]$ will be denoted by $\| \ \|_t$. We shall also make the following assumption.

ASSUMPTION. If $g \in L_1[a, b]$ has the property that $\int_a^b gr\, dx = 0$ for all $r \in R^+$ then $g = 0$ (as an element of L_1).

By a theorem of Cheney and Goldstein [4] this assumption is satisfied by both the polynomial and trigonometric rational families. As a consequence of this assumption we have the following lemma.

LEMMA 1. *Suppose* $f \in C[a, b]$ *is not the zero function and that* $1 < t < \infty$. *Then* 0 *cannot be a best approximation to* f *from* R^+.

Proof. If 0 is best, then the function $\varphi_r(\lambda) = \int_a^b | f + \lambda r |^t\, dx$ has a minimum at $\lambda = 0$ for each $r \in R^+$. But φ_r is differentiable and a direct calculation gives $\varphi_r'(0) = t \int_a^b |f|^{t-1}(\text{sgn } f)r\,dx$. But $\varphi_r'(0) = 0$ for all $r \in R^+$ so that $|f|^{y-1}\,\text{sgn}(f) = 0$ almost everywhere and this clearly implies that $f = 0$ almost everywhere. But f is continuous so $f(x) = 0$ for all x and this is a contradiction. Q.E.D.

LEMMA 2. *Let $f \neq 0$ be continuous on $[a, b]$ and if $t = 1$ assume 0 is not a best approximation to f from R^+. For each v, let g_v be a best approximation to f from $\overline{R([h_v])}$. Then there is a v_0 such that for all $v \geqslant v_0$, g_v is not in the set S_2 of Theorem 2.*

Proof. Assume the lemma is false. Then there is a sequence of subsets $L_j \equiv T_{v_j} \subset [h_{v_j}]$ such that each L_j contains at most m elements and such that $\varphi_j \equiv g_{v_j} = f$ on L_j and vanishes elsewhere. Let $C_j = [h_{v_j}] \sim L_j$ and let r_0 be any element of R^+ such that $\| f \|_t > \| f - r_0 \|_t$. The hypotheses and assumption imply this is possible. Then $\| f - g_v \|_v \leqslant \| f - r_0 \|_v$ for all v and so by continuity of f and $f - r_0$ (and since each L_j has at most m points) we obtain

$$\| f \|_t = \overline{\lim_j}\, \| f - \varphi_j \|_{v_j} \leqslant \overline{\lim_j}\, \| f - r_0 \|_{v_j} = \| f - r_0 \|_t$$

contradicting $\| f \|_t > \| f - r_0 \|_t$. Q.E.D.

Remark. The proof of the following lemma is a simple revision of a similar lemma in [1].

LEMMA 3. *Let $1 \leqslant t \leqslant \infty$ be arbitrary and let $f \in C[a, b]$. Assume that for each v, $g_v \in S_1$ is a best approximation to f with respect to the norm $\| \cdot \|_v$ and let r_v and $S_v \subset [h_v]$ be such that $r_v \equiv p_v/q_v \in R([h_v] \sim S_v)$, $g_v = r_v$ on $[h_v] \sim S_v$, and S_v contains $l_v \leqslant \min(m - 1, n - 1)$ elements. Then there is a set $F \subset [a, b]$ whose complement has Lebesgue measure zero and an element $r \in \hat{R}$ such that for some sequence $\{r_{v_j}\}$ we have $r_{v_j}(x) \to r(x)$ for all $x \in F$. In fact, $F = \bigcup_{j=1}^{\infty} F_j$ where $F_j \subset F_{j+1}$, $j = 1, 2,\ldots$, F_j is a finite union of closed intervals, and $r_{v_j} \to r$ uniformly on each F_j.*

Proof. Let $A_v = [h_v] \sim S_v$. The sequence $\{\| r_v \|_{A_v}\}$ is bounded since $\| f - r_v \|_{A_v} \leqslant \| f - g_v \|_v \leqslant \| f \|_v \to \| f \|_t$ as $v \to \infty$. Moreover, we may assume that $\| q_v \|_\infty = 1$ for all v.

CLAIM. *$\{\| p_v \|_\infty\}$ is bounded.*

Proof. Assume the claim is false and let $r_v' = r_v / \| p_v \|_\infty$. By passing to a subsequence if necessary, we may assume that $\| r_v' \|_{A_v} \to 0$. Let p_v' denote $p_v / \| p_v \|_\infty$. Then we may assume that $p_v' \to p^* \in P$ and $q_v \to q^* \in Q$ uniformly where $\| p^* \|_\infty = \| q^* \|_\infty = 1$. Then $r_v' \to p^*/q^*$ uniformly on each closed subset of the set $\{x \mid q^*(x) \neq 0\}$. Pick a closed subinterval $I \equiv [\alpha, \beta] \subset [a, b]$

such that neither p^* or q^* has a root in I and let B_v denote the set $([h_v] \sim S_v) \cap I$. Then $\inf_{x \in I} |p^*(x)| \equiv \delta > 0$ so that

$$\| r_v' \|_{B_v} \geqslant \left\| \frac{p_v'}{\| q_v \|_\infty} \right\|_{B_v} = \| p_v' \|_{B_v} = \left(\frac{b-q}{N_v} \right)^{1/t} \left[\sum_{x \in B_v} | p_v'(x)| \right]^{1/t}$$

$$\geqslant \left(\frac{(b-a)\,\theta_v}{N_v} \right)^{1/t} \frac{\delta}{2} \quad \text{for all } v \text{ sufficiently large,}$$

where θ_v is the number of elements in B_v. But $\theta_v/N_v \to (\beta - \alpha)/(b-a)$ as $v \to \infty$. Hence for v sufficiently large, $\| r_v' \|_{A_v} \geqslant \| r_v' \|_{B_v} \geqslant (\beta - \alpha)^{1/t}(\delta/4) > 0$ which is a contradiction and so the claim is proved.

Thus, $\{\| p_v \|_\infty\}$ is bounded and so there exist subsequences (which we do not relabel) $\{p_v\}$ and $\{q_v\}$ and polynomials p and q such that $p_v \to p \in P$ and $q_v \to q \in Q$ uniformly where $\| q \|_\infty = 1$. As before, $r_v \equiv p_v/q_v$ converges to $r \equiv p/q$ uniformly on each closed subset of the set $\{x \mid q(x) \neq 0\}$. Now this set can be written as $\bigcup_{j=1}^\infty F_j$ where each F_j is a finite union of closed intervals with $F_j \subset F_{j+1}$ for all j where $r_v \to r$ uniformly on each F_j. Letting $A_{vj} = A_v \cap F_j$ we have $\| r \|_{A_{vj}} \leqslant \| r_v - r \|_{A_{vj}} + \| r_v \|_{A_{vj}}$ so that

$$\overline{\lim_v} \| r \|_{A_{vj}} \leqslant \overline{\lim_v} \| r_v \|_{A_{vj}} + 0 \leqslant \overline{\lim_v} (\| r_v - f \|_{A_{vj}} + \| f \|_{A_{vj}})$$

$$\leqslant \overline{\lim_v} (\| g_v - f \|_v + \| f \|_v) \leqslant \overline{\lim_v} \| g_v - f \|_v + \overline{\lim_v} \| f \|_v$$

$$\leqslant \overline{\lim_v} \| f \|_v + \overline{\lim_v} \| f \|_v = 2 \| f \|_t \quad \text{since } g_v \text{ is a best}$$

approximation to f so that for each v $\| g_v - f \|_v \leqslant \| f \|_v$. Thus, there is a constant $M > 0$ independent of j such that $\overline{\lim}_v \| r \|_{A_{vj}} \leqslant M$ for all j. But

$$\overline{\lim_v} \| r \|_{A_{vj}} = \left[\int_{F_j} | r(x)|^t \, dx \right]^{1/t}$$

for each j since r is continuous on F_j and so $\int_{F_j} | r(x)|^t \, dx \leqslant M^t$ for all j. Since $F \equiv \bigcup_{j=1}^\infty F_j$ has measure $b - a$ and since $F_j \subset F_{j+1}$ we conclude from the monotone convergence theorem that $r \in L_t[a, b]$. But this means that $r \in \hat{R}$ and the proof is complete. Q.E.D.

We are now ready for the main result of this paper.

THEOREM 4. *Let $1 \leqslant t < \infty$ be arbitrary and let $f \in C[a, b]$. Assume that some best approximations to f from $\hat{R}$ is in R^+. For each g_v, let $\{g_v\}$ be a sequence of best approximations to f from $\overline{R([h_v])}$. Then $\{g_v\}$ has a subsequence converging almost everywhere to a best approximation to f from $\hat{R}$. If, in addition, $f \in NP$ and has all its best approximations to f (from $\hat{R}$) in R^+ then* *there is a v_0 such that for all $v \geqslant v_0$, f has a best approximation in $R([h_v])$ and any sequence of such approximations, say $\{r_v\}$, has a subsequence converging uniformly over $[a, b]$ to a best approximation to f from R^+.*

Proof. Let g_v be as above. Then by Lemma 2 we may assume that $g_v \notin S_2$ and so by Lemma 3, some subsequence converges to an element $r \in \hat{R}$ almost everywhere. It remains to show that r is a best approximation to f. Let $r' \in R^+$ be a best approximation to f from $\hat{R}$. For notational convenience we do not relabel the convergent subsequence of $\{g_v\}$. Let the sequence of compact sets $\{F_j\}, j = 1, 2,...$ be as in Lemma 3. That is, $F_j \subset F_{j+1}$ for all j, $g_v \to r$ uniformly on each F_j, F_j is a finite union of closed intervals, and the measure of $(\bigcup_{j=1}^{\infty} F_j)^C$ is zero. Again let A_{vj} denote $F_j \cap ([h_v] \sim S_v)$. Then

$$\|f - r\|_{A_{vj}} - \|g_v - r\|_{A_{vj}} \leqslant \|f - g_v\|_{A_{vj}} \leqslant \|f - g_v\|_v \leqslant \|f - r'\|_v$$

so that

$$\overline{\lim_{v}} \, (\|f - r\|_{A_{vj}} - \|g_v - r\|_{A_{vj}}) \leqslant \overline{\lim_{v}} \, \|g - g_v\|_v \leqslant \overline{\lim_{v}} \, \|f - r'\|_v .$$

But $\|f - r\|_{A_{vj}} \to [\int_{F_j} |f(x) - r(x)|^t \, dt]^{1/t}$, $\|g_v - r\|_{A_{vj}} \to 0$ (since $\|g_v - r\|_{A_{vj}} \leqslant \sup_{x \in A_{vj}} |g_v(x) - r(x)|$), and $\|f - r'\|_v \to \|f - r'\|_t$ as $v \to \infty$. Thus, $[\int_{F_j} |f(x) - r(x)|^t \, dx]^{1/t} \leqslant \|f - r'\|_t$ for all j. But since $(\bigcup_{j=1}^{\infty} F_j)^C$ has measure zero we conclude that $\|f - r\|_t \leqslant \|f - r'\|_t$ and so r is a best approximation.

Now assume that $f \in NP$ and has all its best approximations (from $\hat{R}$) in R^+ and suppose there exists a sequence of best approximations $\{g_v\} \subset S_1$ where $g_v = p_v/q_v \in R([h_v] \sim T_v)$ on $[h_v] \sim T_v$ where T_v has l_v elements with $\min(m - 1, n - 1) \geqslant l_v \geqslant l_0 > 0$. We wish to show that the assumption $l_v \geqslant l_0 > 0$ leads to a contradiction.

By the first part of the theorem some subsequence of $\{g_v\}$ say $\{g_{v_j}\}$ converges almost everywhere to a best approximation $r \in \hat{R}$. Recalling from Theorem 1 that $p_v = q_v = 0$ on the associated set T_v and noting that each q_v has l_0 zero's we have that r can be written in the form p/q where q has at least l_0 roots in $[a, b]$. But since all best approximations to f are in R^+ we conclude that there is an $r' \in R^+$ such that $r' = r$ almost everywhere. Now let $[\alpha, \beta]$ be any subinterval of $[a, b]$ on which $q(x)$ is strictly positive and such that $r = r'$ on $[\alpha, \beta]$. Without loss of generality, we may assume that $\|q\| = 1 = \|q'\|$ where $\|\sigma\| \equiv \sum_{k=1}^{m} |b_k|$ where $\sigma = \sum_{k=1}^{m} b_k \sigma_k$. Since $f \in NP$, r must be normal and so by Theorem 3 there is a unique pair $(p, q) \in P \oplus Q$ such that $r = p/q$ on $[\alpha, \beta]$ and such that $\|q\| = 1$. Thus we have $p' \equiv p$ and $q' \equiv p$ but this is a contradiction since q has roots in $[a, b]$ and q' does not.

Thus the assumption that $l_{v_j} \geqslant l_0 > 0$ for all v leads to contradiction. Thus given any sequence of best approximations $\{g_v\} \subset S_1$ and corresponding sets $\{T_v\}$ no subsequence has the property that the number of elements in the corresponding T_v's (i.e. the l_v's) is bounded away from zero. Since the l_v's are integers we conclude that there is an integer $v_0 \geqslant 0$ such that for all $v \geqslant v_0$, $l_v = 0$ so that $T_v = \varnothing$ and so $g_v \in R([h_v])$ for all such v's. This shows that a best approximation to f exists in $R([h_v])$ for v sufficiently large.

Finally, the uniform convergence of some subsequence to a best approximation follows by observing that in the proof of Lemma 3, the convergence is uniform if the limiting denominator has no roots in $[a, b]$. By normalizing the denominators as in Theorem 3 and using the normality of any best approximation and the fact that all are in R^+ it is clear that the convergence will be uniform over $[a, b]$. Q.E.D.

COROLLARY. *If $f \in NP$ has a unique best approximation r in $\hat{R}$ and it lies in R^+, then any sequence $\{r_v\}$ of best approximations from $R([h_v])$ converges uniformly to r.*

Remark. For the ordinary rational functions or the rational trigonometric functions we always have that $\hat{R} = R^+$ so that the convergence of the corresponding subsequences is always uniform provided that $f \in NP$. One does not always have that $\hat{R} = R^+$ (see [2]) when other rational families are considered. The assumption, however, is satisfied by a wide class of families. For example, if P is any Haar family and $Q = \text{span}\{1, \Phi\}$ (where Φ is continuous and monotone on $[a, b]$) then the assumption is satisfied. For such families Theorem 4 will apply to functions having at least one best approximation in R^+.

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